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PROPAGATION OF SURFACE WAVES THROUGH A STOCHASTIC INHOMOGENEOUS ELASTIC MEDIUM (THE MARKOVIAN APPROXIMATION)

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The problem of behavior of an unsteady surface wave in an inhomogeneous, linearly deformable elastic medium is considered. Investigation of the surface wave fronts is based on the ray representations of the wave, regarded as a line of discontinuity of the displacement derivatives propagating along the boundary surface. A system of partial differential equations is reduced using the methods of the theory of discontinuous solutions together with the dynamic, kinematic and geometrical conditions of compatibility, to an ordinary differential equation in terms of the wave intensity, with the velocity of this wave coinciding at every point of the inhomogeneous surface with the Rayleigh velocity. This equation is supplemented by a system of relations characterizing the change in the geometrical parameters of the surface front in the course of the propagation. Specific models of the stochastic media are considered for which the processes under investigation are Markovian and can be described with help of the methods of the theory of multidimensional Markovian stochastic processes. Conditions are established concerning the character of the distribution of the surface inhomogeneity, which admit the application of the Markovian approximation. When the wave propagate through randomly inhomogeneous media, the presence of free boundaries leads to appearance of a number of the boundary phenomena sufficiently well defined in some boundary zone [1—4]. The appearance of waves propagating along the free surface is connected with the possibility of existence of inhomogeneous waves near the boundary [5—7]. The harmonic surface waves in randomly inhomogeneous media were studied in [8] with help of the approximations of the geometrical optics (short waves), and in [1] within the framework of the method of effective parameters (long waves). An approach based on the use of the Markovian approximations was developed in [9] for investigating the processes of propagation of volume waves.

1. Let an inhomogeneous isotropic medium the elastic moduli of which depend randomly on the coordinates, be bounded by an arbitrary, sufficiently smooth boundary S. Using an x^n-coordinate system attached to the free surface S, we can write the dynamic relationships in the form

$$\nabla \sigma_{ij} - \rho u_i^{\cdot\cdot} = 0, \qquad i = 1, 2, 3 \tag{1.1}$$
$$\sigma_{ij} = \lambda \nabla_k u^k G_{ij} + \mu (\nabla_i u_j + \nabla_j u_i)$$

$$\sigma_{ij}n^j = 0, \quad x^n \in S$$

Here u_i denotes the displacement vector, σ_{ij} is the stress tensor, $\lambda(x^n)$, $\mu(x^n)$, $\rho(x^n)$ are the elastic moduli and the density of the medium, G_{ij} is the metric tensor of the system x^n and is chosen as follows: x^1 and x^2 are the curvilinear coordinates on S, x^3 is the distance along the normal in the inward direction, while ∇^j and ∇_j denote the contravariant and covariant tensor differentiation in the metric of x^n.

The presence of a boundary surface means that inhomogeneous waves may exist in the medium. When we say an inhomogeneous, first order wave, we mean by this a one-parameter family of directed complex surfaces Σ on which the displacements are continuous, but their derivatives may have a discontinuity. The dynamic, kinematic and geometrical conditions of compatibility all hold on Σ. At every point of the medium in question the longitudinal and transverse surfaces of discontinuity $\Sigma^{(k)}$ $(k = 1, 2)$ propagate in the directions of their normals at the rate

$$c_{(1)} = \{(\lambda + 2\mu) / \rho\}^{1/2}, \quad c_{(2)} = (\mu / \rho)^{1/2}$$

Let us derive the conditions needed for the existence of first order surface waves on S. We use the term "surface wave" to describe a one-parameter family of real curves l on which the displacements defined on S are continuous, while their first derivatives have a discontinuity on passing across l. We denote by $[f]_{(k)}$ $(k = 1, 2)$ the jump in the value of the function f at the surface $\Sigma^{(k)}$, and by $[f]_s$ the jump in the value of f at the line l. Here and henceforth the index (1) will refer to the relations on the longitudinal wave, and (2), to relations on the transverse wave. The surface waves are constructed in the form of a linear combination of the inhomogeneous longitudinal and transverse waves, and as a result of this we seek the jump at the surface S in the form of a sum

$$[f]_s = [f]_{(1)} + [f]_{(2)} \tag{1.2}$$

The last two equations of (1.1) yield

$$[\sigma_{ij}]_s n^j = 0$$
$$[\sigma_{ij}]_s = \lambda [\nabla_k u^k]_s G_{ij} + \mu([\nabla_j u_j]_s + [\nabla_j u_i]_s) \tag{1.3}$$

The relations (1.3) with (1.2) and the conditions of the first order compatibility [10]

$$[\nabla_j u_i] = \omega_i \nu_j |_{(k)}, \quad \omega_i^{(k)} = [\nabla_n u_i] \nu^n |_{(k)}$$

where $\nu_i^{(k)}$ are the components of the normal to $\Sigma^{(k)}$, all taken into account, are written in the form

$$\lambda\omega^k\nu_k n^i + \mu(\omega_j\nu^i + \omega^i\nu_j) n^j |_{(1)} + \mu(\omega_j\nu^i + \omega^i\nu_j) n^j |_{(2)} = 0 \tag{1.4}$$

We arrange the x^n-coordinate.system in such a manner, that the normal vector $\mathbf{n}$ has the contravariant components (0, 0, 1) and choose the parametric net on S so that the coordinate x^1 determines the time of propagation along the surface ray, and x^2 characterizes a point on l. Then at the points belonging to S we have

$$\nu_2^{(k)} = \nu^{2(k)} = 0 \tag{1.5}$$

On the transverse wave we have $\omega_i \nu^i |_{(2)} = 0$, consequently for the covariant and contravariant components of the wave vector we obtain

$$\omega_1 = -kv_3^{(2)}c, \quad \omega_3 = kv_1^{(2)} / c \qquad (1.6)$$
$$\omega^1 = -kv_{(2)}^3 / c, \quad \omega^3 = kv_{(2)}^1 c, \quad k = (\omega_i \omega^i)_{(2)}^{1/2}$$

On the longitudinal wave we have

$$\omega_i = \omega v_i^{(1)}, \quad \omega^i = \omega v_{(1)}^i, \quad \omega = (\omega_i \omega^i)_{(1)}^{1/2} \qquad (1.7)$$

where k and ω denote the intensities of the volume waves, determined as the moduli of the wave vectors $\omega_i^{(k)}$.

In the course of deriving (1.6), we assumed that at the points belonging to S

$$G_{11} = c^2, \quad G_{33} = 1, \quad G_{13} = 0$$

where c is the rate of propagation of the front l along the ray on S.

Using (1.5)–(1.7) we transform (1.4) to the form

$$2\omega (v^1 v_3)_{(1)} + k (v^1 v_1 - v^3 v_3)_{(2)} / c = 0 \qquad (1.8)$$
$$\omega (\lambda + 2\mu v^3 v_3)_{(1)} + 2k\mu (v^3 v_1)_{(2)} / c = 0$$

The physical components $N_{i(k)}$ and the tensor components of the normal vector $v_{(k)}$ are connected by the following relations:

$$v_{(k)}^i = N_{i(k)} (G_{ii})^{-1/2}, \quad v_{i(k)} = N_{i(k)} (G_{ii})^{1/2}, \quad k = 1, 2 \qquad (1.9)$$
$$N_{1(k)} = c_{(1)} / c, \quad N_{3(k)} = i \{(c_{(1)} / c)^2 - 1\}^{1/2}, \quad N_{2(k)} = 0, \quad c < c_{(k)}$$

Taking (1.9) into account, we can write the conditions of existence of nonzero solutions of (1.8) in the form

$$\left(\frac{1}{c_{(2)}^2} - \frac{2}{c^2}\right)^2 - \frac{4}{c^2}\left(\frac{c_{(1)}^2}{c^2} - 1\right)^{1/2}\left(\frac{c_{(2)}^2}{c^2} - 1\right)^{1/2} = 0 \qquad (1.10)$$

Equations (1.10) which determines the velocity c at every point of the surface, coincides with the Rayleigh equation. An analogous result was obtained in [6] for an inhomogeneous medium where the representations of the high frequency asymptotics were used in somewhat different form.

Let us write the general solution of the system (1.8) with condition (1.10), in the form

$$\omega = -(v^1 v_1 - v^3 v_3)_{(2)} \chi / c, \quad k = 2 (v^1 v_3)_{(2)} \chi \qquad (1.11)$$

and call the quantity χ, which completely defines the discontinuity at the boundary, the intensity of the surface wave.

Let us see how χ varies along the surface ray. To do this, we use the second order dynamic conditions of compatibility as well as the kinematic and geometrical conditions at the surfaces $\Sigma^{(k)}$ given in the parametric form by $x^i = x^i (y_{(k)}^1, y_{(k)}^2, t)$

$$[\sigma_{ij}^{\cdot}]_s n^j = 0, \quad [\sigma_{ij}^{\cdot}]_s = \sum_{k=1}^{2} [\sigma_{ij}^{\cdot}]_{(k)} \qquad (1.12)$$
$$[\sigma_{ij}^{\cdot}]_{(k)} = \lambda [\nabla_n u^{\cdot n}]_{(k)} G_{ij} + \mu ([\nabla_i u_j^{\cdot}]_{(k)} + [\nabla_j u_i^{\cdot}]_{(k)})$$
$$[\nabla^j \sigma_{ij}]_{(k)} - \rho [u_i^{\cdot\cdot}]_{(k)} = 0 \qquad (1.13)$$
$$[\nabla^j \sigma_{ij}]_{(k)} = \lambda [\nabla^j \nabla_n u^n]_{(k)} G_{ij} + \mu ([\nabla^j \nabla_i u_j]_{(k)} + [\nabla^j \nabla_j u_i]_{(k)})$$

$$[\nabla_i u^{\cdot}_j]_{(k)} = (-W_i c_{(k)} + D\omega_i / Dt)\,\nu_j - c_{(k)} g^{\alpha\beta}\omega_{i,\,\alpha} x_{j\beta} - g^{\alpha\beta} x_{j\beta}\omega_i c_{,\alpha}^{(k)}|_{(k)} \qquad (1.14)$$
$$[u_i^{\cdot\cdot}]_{(k)} = W_i c_{(k)}^2 - 2c_{(k)} D\omega_i / Dt - \omega_i Dc_{(k)} / Dt\,|_{(k)}$$
$$[\nabla^j \nabla_k u_i] = W_i \nu^j \nu_k + g^{\alpha\beta}\omega_{i,\,\alpha}(\nu^j x_{k\beta} + \nu_k x_\beta{}^j) - \omega_i g^{\alpha\beta} g^{\sigma\tau} x_\beta{}^j x_{k\tau} x_{\alpha\sigma}^n \nu_n\,|_{(k)}$$
$$x_\beta{}^i = \partial x^i / \partial y^\beta, \quad x_{i\beta} = G_{ik} x_\beta{}^k, \quad g_{\alpha\beta} = x_\alpha{}^i x_{i\beta}$$

Here the greek indices assume the values 1 and 2, the latin indices 1, 2 and 3, and the derivative $D / D|_{(k)}$ is defined in [10].

Relations (1.13) and (1.14) yield the expressions for the characteristic second order quantities $W_i^{(k)} = [\nabla_n \nabla_j u_i]\ \nu^n \nu^j$, and the transport equation for the volume waves

$$W_i = W_n \nu^n \nu_i + \omega_{,\alpha} g^{\alpha\beta} x_{i\beta} - 2\rho c_{(1)} g^{\alpha\beta} x_{i\beta} c_{(1),\alpha}\omega / (\lambda + \mu)\,|_{(1)} \qquad (1.15)$$
$$W_n \nu^n = -\omega_{,\alpha}^n g^{\alpha\beta} x_{n\beta} - 2\rho c_{(2)} g^{\alpha\beta} x_\beta{}^n c_{(2),\alpha}\omega_n / (\lambda + \mu)\,|_{(2)} - \mu_{,n}\omega_{(2)}^n / (\lambda + \mu)$$
$$D\omega_i / Dt - c_{(k)}\Omega\omega_i + \omega_i D \ln c_{(k)} / 2Dt = 0 \qquad (1.16)$$

Substituting (1.14) ainto (1.12) and taking into account (1.15), we obtain a system of inhomogeneous equations in $W_{(1)} = (W_i W^i)_{(1)}^{1/2}$, $W_{(2)} = (W_i W^i)_{(2)}^{1/2}$, with a determinant which is equal to zero by virtue of (1.10).

The necessary and sufficient condition of existence of a solution of this system has the form

$$2Q_1(\omega, k)(\nu^3\nu_1)_{(2)} - Q_2(\omega, k)(\nu^1\nu_1 - \nu^3\nu_3)_{(2)} = 0 \qquad (1.17)$$
$$Q_1 = B^1 + (x_{3\beta}\nu^1 + x_\beta{}^1\nu^3)_{(1)} K^\beta + \nu_{3(2)} I / \nu_{1(2)}$$
$$Q_2 = B^3 + 2(x_{3\beta}\nu^3)_{(1)} K^\beta + \lambda I$$
$$B^i = n^j \sum_{k=1}^{2} \{\lambda G_j{}^i L_n{}^n + \mu(L_j{}^i + G_k{}^i G_j{}^n L_n{}^k)\}_{(k)}$$
$$L_j{}^i = (D\omega^i / Dt)\,\nu_j - g^{\alpha\beta} c_{(k)}\omega_{,\alpha}{}^i x_{j\beta}$$
$$K^\beta = \mu c_{(1)} g^{\alpha\beta}\{\omega_{,\alpha} + (\lambda + 2\mu)_{,\alpha}\,\omega / (\lambda + \mu)\}_{(1)}$$
$$I = \mu c_{(2)}\{g^{\alpha\beta} x_\beta{}^n(\omega_{n,\,\alpha} + \mu_{,\alpha}\omega_n) / (\lambda + \mu) - \mu_{,n}\omega^n / (\lambda + \mu)\}_{(2)}$$

Passing now in the expressions for Q_1 and Q_2 from the coordinates $\overset{1}{y}_{(k)}$, $\overset{2}{y}_{(k)}$ and the derivative $D / Dt\,|_{(k)}$, associated with the surfaces $\Sigma_{(k)}$, to a single coordinate system governed by the geometry of the curve l and surface S, and replacing ω and k by χ, with (1.11) and (1.16) taken into account, we obtain the following equation for $X = |\chi|$ from (1.17):

$$\frac{dX}{ds} + \frac{1}{4}\frac{d\ln g_{22}^{(s)}}{ds} X + \frac{1}{2}\frac{d\ln R}{ds} X = 0 \qquad (1.18)$$
$$R = c^{-4} N_{(2)1}^{-5} N_{(1)1}^{-2}(1 - N_{(2)1}^2)^{-1/2}\{-6N_{(1)1}^2 N_{(2)3}^2(1 + N_{(2)1}^2) + 4N_{(2)1}^2 N_{(2)3}^2 + 2N_{(2)1}^2(N_{(1)1}^2 - 2N_{(2)1}^4)\}$$

Here $d / ds = c^{-1} D / Dt$ is the derivative along the surface ray and $g_{22}^{(s)}$ is the corresponding component of the first quadratic form of S characterized by the geometry of the surface wave front (divergence of the surface rays).

Let us write the solution of (1.18) in the form

$$\mathrm{X} = \mathrm{X}_0 (g_{22}^{(s)0} / g_{22}^{(s)})^{1/4} (R_0 / R)^{1/2}, \quad g_{22}^{(s)0} = g_{22}^{(s)}(0), \quad R_0 = R(0) \tag{1.19}$$

Suppose that S is a plane. In this case we have

$$-d \ln g_{22}^{(s)} / ds = 4\Omega, \quad d\Omega / ds = 2\Omega^2 + g^{22} c_{,22} / c \tag{1.20}$$
$$c_{,22} / c = (\lg c)_{,22} + (\ln c)_{,2}{}^2 = N(s)$$

where Ω denotes the mean curvature of the cylindrical surface with the directrix l (see [9]).

The system (1.18), (1.20) is closed and determines the solution $\mathrm{X}(s)$, $\Omega(s)$, $g_{22}(s)$ for the given initial conditions and velocity $c(s)$, in a unique manner.

Introducting the variables

$$\zeta_2 = (\mathrm{X}/ \mathrm{X}_0)^4 (R / R_0)^2, \quad \zeta_1 = d\zeta_2 / ds \tag{1.21}$$

we obtain the equation for ζ_1 in the form

$$\frac{d\zeta_1}{ds} = \frac{3\zeta_1}{2\zeta_2} + 4N, \quad \zeta_2(0) = 1, \quad \zeta_1(0) = \zeta_1{}^\circ \tag{1.22}$$

The system of equations (1.21), (1.22) enables us to determine the surface wave intensity as a function of the length of the ray s. The coordinates of the points lying on the ray satisfy the relations

$$d\mathbf{r} / ds = \mathbf{v}, \quad dv_i / ds = -g^{22} x_{i2} (\ln c)_{,2} \tag{1.23}$$

where $\mathbf{r} = \{x_1, x_2\}$ is the radius vector of the points on S, x_1 and x_2 are the Cartesian coordinates of the surface, and $\mathbf{v}$ is the normal to l in the plane S.

2. Since the relations (1.18), (1.20) are not linear, the stochastic approach leads to an infinite system of interlinked momentum equations which require a specified procedure for closing [2, 4, 11]. Let us consider the problem of using the Markovian approximations to describe the process of propagation of a surface wave. We make use of the equations (1.21), (1.22), writing them in the form

$$d\xi_i / ds = \Phi_i(\xi_j, \eta_j), \quad i = 1, 2, \ldots, n_1 \tag{2.1}$$

where $\eta_j(s)$ denote the nonlinear functions of $c(s)$, $c_{(1)}(s)$, $c_{(2)}(s)$. In the present case these functions are $N(s)$ or $\ln R$. We assume that $\eta_j(s)$ are steady random functions with rational fractional spectral densities. In this case $\eta_j(s)$ satisfy the equations

$$d\eta_j / ds = F_j(\eta_k) + G_j(\eta_k) q_k, \quad j = 1, 2, \ldots, n_2 \tag{2.2}$$
$$\langle q_k(s) \rangle = 0, \quad \langle q_k(s) q_n(s') \rangle = \delta(s - s') A_{kn} / \pi$$

where F_j and G_j are arbitrary functions, and $q_k(s)$ is "white noise".

Relations (2.2) which are used to define the model of a stochastically inhomogeneous medium for a particular choice of the functions F_j and G_j can be called, in analogy with [12], the equation of "shaping filter". We use this term here in the sense that every shaped model filters out some definite frequencies from the perturbations propagating through it. If $F_j \equiv 0$, $G_j \equiv 1$, then $\eta_j(s)$ is a normal Wiener process [12].

Increasing the number of the variables ξ_i on account of the functions $\eta_i(s)$, we shall write the relations (2.1), (2.2) in the form

$$d\xi_m / ds = h_m(\xi_k) + f_m(q_k, \xi_k), \quad m = 1, 2, \ldots, n_1 + n_2 \tag{2.3}$$

For the equations (1.21) and (1.22) and the Wiener model

$$d\zeta_3 / ds = q(s), \quad \zeta_3 = 4N \tag{2.4}$$

the components of the vector functions ξ_i, h_i and f_i are

$$\boldsymbol{\xi} = \{\zeta_1, \zeta_2, \zeta_3\}, \quad \mathbf{h} = \left\{\frac{3\zeta_1^2}{2\zeta_2} + \zeta_3, \zeta_1, 0\right\} \qquad \mathbf{f} = \{0, 0, q(s)\} \tag{2.5}$$

In deriving the system (2.3), (2.5) we made the assumption that the nonlinear function N can be δ-correlated with the elastic velocities $c(s)$, $c_{(1)}(s)$ and $c_{(2)}(s)$ along the ray. As we know [2, 12], the random quantities satisfying the first order differential equations of the type (2.3) are Markovian, and the probability distribution of these quantities is described by the Fokker-Plank-Kolmogorov (FPK) equation.

The FPK equation corresponding to the system (1.22), (2.4) will assume the form ($P(\zeta_i, s)$ is the distribution density of the probabilities $\zeta_i(s)$ at the point s)

$$\frac{\partial P}{\partial s} + 3\frac{\zeta_1}{\zeta_2}P + \left(\zeta_3 + \frac{3\zeta_1^2}{2\zeta_2}\right)\frac{\partial P}{\partial \zeta_1} + \zeta_1\frac{\partial P}{\partial \zeta_2} - A\frac{\partial^2 P}{\partial \zeta_3^2} = 0 \tag{2.6}$$

In the same manner we can obtain stochastic differential equations of the type (2.3) from the relations (1.18), (1.20) and (1.23), and write the density distribution equation for this system.

Let us inspect some cases in which the solutions of the FPK equation are simple. Let the elastic properties of the medium vary only in the direction of the ray on S, and remain constant along the front l. Then

$$\Omega(s) = \Omega_0 \varkappa(s), \quad \Omega_0 = {}^1\!/_2\Omega_{0l}, \ \varkappa(s) = (1 - 2\Omega_0 s)^{-1}$$

where Ω_{0l} denotes the curvature of l at the initial instant.

We consider two models of the stochastically inhomogeneous medium

$$1)\ d(\ln R) / ds = q(s); \qquad 2)\ d(\ln R) / ds = \ln R + q(s)$$

For the model 1) the logarithm of intensity satisfies the stochastic equation

$$dH / ds = \Omega_0\varkappa(s) + q(s), \quad H = \ln X$$

and the solution of the corresponding equation for the distribution density has the form

$$P(H, s) = \sqrt{\frac{1}{4\pi As}} \exp\left\{-\frac{H - H_0 - {}^1\!/_2 \ln \varkappa(s)}{4As}\right\} \tag{2.7}$$

The longitudinal correlation H is calculated as in [9], and this gives

$$\langle H(s) H(s')\rangle = 2As' + [H_0 + {}^1\!/_2 \ln \varkappa(s')]^2 - {}^1\!/_2 [H_0 + {}^1\!/_2 \ln \varkappa(s')] [\ln \varkappa(s') - \ln \varkappa(s)], \quad (s > s')$$

By virtue of (2.7), the one-dimensional distribution density $P^{(1)}(X, s)$ can be written in the form

$$p^{(1)}(X, s) = \frac{X_0}{2X\sqrt{\pi A s}} \exp\left\{-\frac{[\ln(X/X_0) - {}^1/_2 \ln \varkappa(s)]^2}{4As}\right\}$$

For a model of the medium described by condition 2), we obtain a similar expression in which the quantity $2As$ must be replaced by $\sigma^2(s)$ and

$$\sigma^2(s) = {}^1/_2\,[1 - \exp(-4As)]$$

We note that in the case of model 2) the intensity of the plane wave has the following steady state distribution:

$$p^{(2)} = \frac{X_0}{X\sqrt{\pi}} \exp\left\{\left(\ln\frac{X}{X_0}\right)^2\right\}$$

We can see that the intensity of the surface front has a logarithmic-normal distribution just as in the case of the volume waves [9]. sStatistical observations of the wave intensities in a stochastic medium confirm the feasibility of the models under investigation. The defining equations (2.2) contain all classes of media for which the distribution densities of certain functions of random velocities $c(s)$, $c_1(s)$ and $c_2(s)$ can be described by the Pearson curves [12].

Construction of the corresponding models and use of the Markovian approximations is based on the introduction of a small parameter expressing the relationship connecting the scale of variation of certain quantities depending on the elastic coefficients, with the characteristic dimensions of the dynamic and geometrical parameters of the problem. This condition is expressed mathematically in the assumption of the δ-correlation properties of the functions of the elastic moduli along the ray. The distribution of the elastic parameters is, in this case, close to the logarithmic-normal which occurs e.g. in the case of rocks. Under the assumptions made, the influence of the statistical properties of the inhomogeneity of the medium across the ray was excluded. In the case of a stochastic surface S, the statisticl dependence on the coordinates of S must also be taken into account.

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